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ON THE MATHEMATICAL DESCRIPTION OF SPIRAL WAVES IN DISTRIBUTED CHEMICAL SYSTEMS*

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Selfexcited oscillatory modes in a chemically active medium of general form with diffusion are studied. The reactor is in the shape of a circle with impermeable boundaries and the medium is in mechanical equilibrium. Asymptotic forms are found for the case of a near-threshold value of the parameter for two kinds of selfexcited oscillations, rotating waves and standing symmetric waves, under the assumption that a vibrational loss of chemical equilibrium stability occurs.

Rotating spiral (reverberator) and divergent concentric (donducting centre) waves of chemical concentrations or electrical excitation have been observed in experiments on vibrational modes in distributed biological and chemical systems /1—3/. The reverberator can have several branches (several spiral wave fronts rotate around one local section of the medium). Analogous modes are detected in the combustion of cylindrical specimens /4/. Different approaches (see /5—8/ and the bibliography presented there) were used to describe such modes on the basis of diffusion equations with non-linear kinetics.** In one of the approaches, the occurence of rotating waves was associated with loss of stability of the stationary spatially homogeneous mode, and therefore, was examined from the aspect of the theory of bifurcation of solutions of non-linear equations dependent on a parameter. The analytical difficulties that occur here were successfully overcome in /7, 8/ by using group methods of bifurcation theory /9/. It was found that in the bifurcation situation examined, solutions, periodic in time,

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of just two kinds bifurcate from the stationary solution (which can be considered identically zero) in the general case in the system phase space, namely waves rotating in opposite directions (two such solutions), and standing waves /8/.

However, the method utilized in /7/ did not result in explicit expressions for the asymptotic forms of the bifurcating solutions and for the quantities governing the nature of the bifurcation. To eliminate these deficiencies, a method is used below that stems from the work of V.I. Iudovich /10/ and others. Solutions of a special kind that are periodic in time are sought: solutions that are stationary in a certain rotating coordinate system, and solutions that are even in the angular coordinate. This reduces the symmetry of the initial problem and enables the bifurcation to be investigated by the Liapunov-Schmidt method /11/. The occurrence of solutions of both the types listed is studied, where the fact of standing-wave bifurcation is deduced from the results obtained in /12/.

1. Formulation of the problem. Consider the system of equations

$$\partial \mathbf{u}/\partial t' = D\Delta\mathbf{u} + \mathbf{f}(\mathbf{u},\gamma);\quad \partial\mathbf{u}/\partial\mathbf{v}\,|_{\partial S_R} = 0 \tag{1.1}$$
$$\mathbf{u} = (u_1,\ldots,u_n);\ n\geqslant 3;\ D = \mathrm{diag}\,(d_1,\ldots,d_n)$$
$$d_i > 0,\ i = 1,\ldots,n;\ \mathbf{f}(\mathbf{u},\gamma) = (f_1(\mathbf{u},\gamma),\ldots,f_n(\mathbf{u},\gamma))$$

in a circle S_R of radius R.

Here Δ is the two-dimensional Laplace operator, $\mathbf{v}$ is the normal to the boundary ∂S_R of the circle S_R, and γ is a real parameter. Let r', θ' be polar coordinates in S_R. By means of the substitution $r\to r'/R$, $t\to t'/R^2$ we can reduce (1.1) to the form

$$\partial\mathbf{u}/\partial t = D\Delta\mathbf{u} + \mathbf{F}(\mathbf{u},\gamma)\ \text{ in } S_1\equiv S\quad (\mathbf{F}(\cdot,\gamma) = R^2\mathbf{f}(\cdot,\gamma)) \tag{1.2}$$
$$\partial\mathbf{u}/\partial\mathbf{v} = 0\ \text{ on }\ \partial S \tag{1.3}$$

We note that the role of the parameter γ in (1.2) can be performed by R^2 in particular (here $\mathbf{f}$ is independent of γ).

We assume that the following is satisfied.

Condition 1. The functions $F_i(\mathbf{z},\gamma):\mathbf{R}^n\times\mathbf{R}^1\to\mathbf{R}^1$ are analytic in $\mathbf{z}\in U$ and $\gamma\in\Gamma$ (U is a neighbourhood of zero of the n-dimensional Euclidean $\mathbf{R}^n$, $\Gamma\subset\mathbf{R}^1$ is a certain interval); the Taylor expansion of the vector function $\mathbf{F}(\mathbf{z},\gamma)$ at the zero of $\mathbf{R}^n$ has the form

$$\mathbf{F}(\mathbf{z},\gamma) = \sum_{j=1}^{\infty} M_j(\gamma)\,\mathbf{z}^j;\qquad M_j(\gamma):\mathbf{R}^n\times\ldots\times\mathbf{R}^n\,(j\ \text{раз})\to\mathbf{R}^n \tag{1.4}$$

Here $M_j(\gamma)$ are j-linear operators; all the eigenvalues of the matrix $M_1(\gamma) = \|\partial F_i(0,\gamma)/\partial z_j\|$ lie strictly in the left half-plane of the complex plane.

It follows from condition 1 that the concentrated system corresponding to (1.2)

$$d\mathbf{u}/dt = \mathbf{F}(\mathbf{u},\gamma)\quad (\mathbf{u}\in\mathbf{R}^n,\ \gamma\in\Gamma) \tag{1.5}$$

has the coarse asymptotically stable singularity $\mathbf{u} = \mathbf{0}$. The question of the stability of the zeroth solution of the distributed system (1.2) and (1.3) is related in a known manner to the study of the eigenvalue problem

$$D\Delta\mathbf{v} + M_1(\gamma)\,\mathbf{v} = \sigma\mathbf{v};\quad \partial\mathbf{v}/\partial\mathbf{v}\,|_{\partial S} = 0 \tag{1.6}$$

Each eigenvalue σ of problem (1.6) is an eigenvalue of the matrix $G(\varkappa_{pk}^2,\gamma) = -\varkappa_{pk}^2 D + M_1(\gamma)$, where $\varkappa_{pk}^2$ is a certain eigenvalue of the problem

$$-\Delta\psi = \varkappa^2\psi;\quad \partial\psi/\partial\mathbf{v}\,|_{\partial S} = 0 \tag{1.7}$$

We shall say that the eigenvalue σ of problem (1.6) is generated by a corresponding eigenvalue of problem (1.7). Each eigenvector function of problem (1.6) corresponding to σ has the form

$$\mathbf{v}_\sigma = \mathbf{g}_\sigma\psi_{\varkappa^2} \tag{1.8}$$

$$\mathrm{const}\cdot\psi_{\varkappa^2} = \begin{cases} 1,\ \varkappa^2 = \varkappa_{01}^2 = 0\\ J_0(\varkappa_{0k}r),\ \varkappa^2 = \varkappa_{0k}^2 > 0,\ k > 1\\ J_p(\varkappa_{pk}r)\begin{Bmatrix}\sin p\theta'\\ \cos p\theta'\end{Bmatrix},\ \varkappa^2 = \varkappa_{pk}^2 > 0;\ p > 0;\ k > 1\end{cases} \tag{1.9}$$

or is a linear combination of vector functions of the form (1.8). Here g_σ is an eigenvector of the matrix $G(\varkappa^2,\gamma)$ corresponding to its eigenvalue σ, $\psi_{\varkappa^2}$ is an eigenfunction of problem (1.7) corresponding to the eigenvalue $\varkappa^2$, and $J_p(\cdot)$ is a Bessel function of the first kind of order p. All these known facts are established by expanding the eigenvector functions of problem (1.6) in Fourier series in eigenfunctions of problem (1.7).

For certain values of $\gamma\in\Gamma$ let the trivial solution of system (1.2), (1.3) be asymptotically stable, and for a further change $\gamma \in \Gamma$ at the time $\gamma = \gamma_*$ becomes unstable because of passage of a pair of complex-conjugate eigenvalues $\sigma_{1,2}$ of problem (1.6) into the right half-plane. We will examine the general case, i.e., we will assume the following to be satisfied.

Condition 2. The intersection of the spectrum of problem (1.6) for $\gamma = \gamma_*$ with the imaginary axis consists of a pair of eigenvalues $\sigma_{1,2} = \pm i\omega_0$ $(\omega_0 \neq 0)$; the remaining eigenvalues of problem (1.6) for $\gamma = \gamma_*$ lie strictly in the left half-plane of the complex plane; the $\sigma_{1,2}$ are generated by one and only one eigenvalue $\varkappa_{pk}^2 > 0$ of problem (1.7); the eigenvalues $\pm i\omega_0$ of the matrix $G(\varkappa_{pk}, \gamma_*)$ are single.

Note that when $n = 2$ the situation mentioned is not realized (the spur of the matrix $G(\varkappa_{pk}^2, \gamma_*)$ cannot vanish because of condition 1).

2. Bifurcation of solutions of the rotating spiral wave type.

Below, p and k are non-negative integers defined by condition 2. It is assumed that $p \geqslant 1$. It will be proved that as γ changes a pair of solutions that are periodic in time bifurcate at the critical time from a trivial solution of system (1.2), (1.3), each stationarily in a certain rotating coordinate system and periodically with period $2\pi/p$ in the angular coordinate.

We make the substitution $\theta \to \theta' - \Omega t; \Omega \neq 0$ in (1.2) and we consider the stationary problem for the vector equation obtained

$$M\mathbf{u} \equiv D\Delta\mathbf{u} + \Omega\partial\mathbf{u}/\partial\theta + \mathbf{F}(\mathbf{u}, \gamma) = 0 \tag{2.1}$$

$$\partial\mathbf{u}/\partial r\,|_{r=1} = 0, \qquad \mathbf{u}(r, \theta + 2\pi) = \mathbf{u}(r, \theta) \tag{2.2}$$

We let Q be the complex hull of a set of real-valued vector functions $\mathbf{u} \in C^2(S) \cap C^1(\bar{S})$, satisfying (2.2), and H, H_2 and H_1 the Hilbert spaces obtained by closing Q in the metrics $((\mathbf{u}, \mathbf{v}) \equiv \mathbf{u}\mathbf{v})$

$$(\mathbf{u}, \mathbf{v})_H = \int_S (\mathbf{u}, \mathbf{v})\, dS; \qquad (\mathbf{u}, \mathbf{v})_{H_2} = \int_S [(\mathbf{u}, \mathbf{v}) + (D\Delta\mathbf{u}, D\Delta\mathbf{v})]\, dS \tag{2.3}$$

$$(\mathbf{u}, \mathbf{v})_{H_1} = \int_S \left[(\mathbf{u}, \mathbf{v}) + \left(D\frac{\partial\mathbf{u}}{\partial r}, \frac{\partial\mathbf{v}}{\partial r}\right) + \frac{1}{r^2}\left(D\frac{\partial\mathbf{u}}{\partial\theta}, \frac{\partial\mathbf{v}}{\partial\theta}\right)\right] dS \tag{2.4}$$

Let $T_a : H \to H$ be the representation of a group of rotations of a circle

$$T_a\mathbf{u}(r, \theta) = \mathbf{u}(r, \theta + a) \qquad (\mathbf{u} \in H, a \in \mathbf{R}^1) \tag{2.5}$$

The identity satisfied formally $\quad T_a M\mathbf{u} = MT_a\mathbf{u}$

shows that in addition to each solution $\mathbf{u} \in H_2$; $\partial\mathbf{u}/\partial\theta \neq 0$ in H, Eq.(2.1) allows of a one-parameter family of solutions of the form $\mathbf{u}_a = T_a\mathbf{u}$ for the same value $\Omega \neq 0$. We call the corresponding value $\Omega \neq 0$ the angular frequency. Evidently, a time-periodic solution $\mathbf{u}(r, \theta' - \Omega t)$ of (1.2), which we shall call a solution of the rotating spiral wave (RSW) type, is in one-to-one correspondence with each family of solutions $\mathbf{u}_a \in H_2$; $\partial\mathbf{u}_a/\partial\theta \neq 0$ in H of (2.1) with angular frequency Ω. If the solution $\mathbf{u} \in H_2$; $\partial\mathbf{u}/\partial\theta \neq 0$ in H exists and is periodic in θ with minimal period $2\pi/m$ (m is a natural number), then the RSW type solution of (1.2) possesses the same property. In this case we will call the number m the number of branches of the RSW type solution.

We say that the number γ_* is a branch point of the solution of RSW type if one-parameter families $\Omega^{(\gamma)}$, $\mathbf{u}^{(\gamma)}$: $\mathbf{u}^{(\gamma)} \in H_2$; $\partial\mathbf{u}^{(\gamma)}/\partial\theta \neq 0$ in H $\gamma \neq \gamma_*$ exist continuous in $\gamma \in \Gamma$ and satisfying (2.1) when γ runs through a certain interval having γ_* as its limit point, where $\Omega^{(\gamma)} \to \omega_0/m$, $\mathbf{u}^{(\gamma)} \to 0$ in H_2 for $\gamma \to \gamma_*$. Together with each pair $\mathbf{u}, \Omega$: $\mathbf{u} \in H_2$; $\partial\mathbf{u}/\partial\theta \neq 0$ in H; $\Omega \neq 0$ satisfying (2.1), the pair $\mathbf{v}, -\Omega$ satisfies this equation, where $\mathbf{v}(r, \theta) \equiv \mathbf{u}(r, -\theta)$.

Let the Taylor expansion of the vector-function $\mathbf{F}(\mathbf{z}, \gamma)$ in the neighbourhood of the point $(0, \gamma_*) \in \mathbf{R}^n \times \mathbf{R}^1$ have the form

$$\mathbf{F}(\mathbf{z}, \gamma) = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} \delta^l M_{jl}\mathbf{z}^j; \qquad \delta = \gamma - \gamma_* \tag{2.6}$$

We introduce the operator $B_\Omega : H_2 \to H$ by setting for each vector function $\mathbf{u} \in H_2$

$$B_\Omega\mathbf{u} \equiv D\Delta\mathbf{u} + \Omega\partial\mathbf{u}/\partial\theta + M_{10}\mathbf{u} \tag{2.7}$$

It can be shown by a Fourier series expansion that when condition 2 is satisfied the operator B_Ω has a zero eigenvalue if and only if $\Omega = \pm\Omega_0$, $\Omega_0 = \omega_0/p$. The kernel $N(B)$ of the operator $B_{\Omega_0} = B$ is two-dimensional; we select the following vectors as basis in $N(B)$

$$\varphi_1 = \mathbf{g}e^{-ip\theta}J_p(\varkappa_{pk}r); \qquad \varphi_2 = \bar{\mathbf{g}}e^{ip\theta}J_p(\varkappa_{pk}r) \tag{2.8}$$

($\mathbf{g}$ is the eigenvector of the matrix $G(\varkappa_{pk}^2, \gamma_*)$ its corresponding eigenvalue is $+i\omega_0$). It can be shown similarly that the operator $B_\Omega^*: H_2 \to H$ conjugate to B_Ω in H

$$B_\Omega{}^*\mathbf{u} \equiv D\Delta\mathbf{u} - \Omega\partial\mathbf{u}/\partial\theta + M_{10}{}^T\mathbf{u} \tag{2.9}$$

($(\cdot)^T$ denotes transposition) has a zero eigennumber if and only if $\Omega = \pm\Omega_0$; the kernel $N(B^*)$ of the operator $B_{\Omega_0}{}^* = B^*$ is two-dimensional and has the following vectors as basis

$$\Phi_1 = \mathbf{g}^T e^{-ip\theta} J_p(\varkappa_{pk} r); \qquad \Phi_2 = \overline{\mathbf{g}^T} e^{ip\theta} J_p(\varkappa_{pk} r) \tag{2.10}$$

where $\mathbf{g}^T$ is the eigenvector of the matrix $G^T(\varkappa_{pk}{}^2, \gamma_*)$ corresponding to its eigenvalue $-i\omega_0$. It follows from condition 2 that $(\mathbf{g}, \mathbf{g}^T) \neq 0$. We take the normalization condition

$$(\mathbf{g}, \mathbf{g}^T) = (2\pi I_{pk})^{-1}; \qquad I_{pk} = \int_0^1 J_p{}^2(\varkappa_{pk} r)\, r\, dr \tag{2.11}$$

such that $(\varphi_i, \Phi_j)_H = \delta_{ij}$; $i, j = 1, 2$.

Lemma 1. For the operator B_Ω to be reversible, it is necessary and sufficient that the points $\pm ip\Omega$ do not belong to the spectrum of the problem (1.6) for $\gamma = \gamma_*$ (the number p is defined in condition 2).

Lemma 2. Only critical values of the parameter γ can be bifurcation points of the RSW types solution: let $\gamma \to \gamma_1$, $\Omega^{(\gamma)} \to \Omega_1$, and let the corresponding non-trivial solutions $\mathbf{u}$ of (2.1) tend to zero in the norm H_2. Then the numbers $\pm ip\Omega_1$ belong to the spectrum of problem (1.6) for $\gamma = \gamma_1$.

Lemma 3. For the equation

$$B\mathbf{u} = \mathbf{h} \quad (\mathbf{h} \in H) \tag{2.12}$$

to be solvable, it is necessary and sufficient that the conditions

$$(\mathbf{h}, \Phi_i)_H = 0, \quad (i = 1, 2) \tag{2.13}$$

be satisfied.

The proof of necessity in Lemma 3 is trivial. To prove the sufficiency in Lemma 3 and Lemma 1, the equation $B_\Omega\mathbf{u} = \mathbf{h}$ $(\mathbf{h} \in H)$ is reduced to an equation with a fully continuous operator in H_1, after which Fredholm's theorem is applied. Such a reduction can be performed by the method described in /13/, say. Every solution $\mathbf{u} \in H_1$ of the equation obtained is an element from H_2 as can be seen by applying Riesz's theorem. Lemma 2 is proved in exactly the same way as the proof of Lemma 1.3 in /12/; here the Hölder and Minkowski inequalities and the imbedding inequalities

$$\max_{(r,\theta)\in S} |\mathbf{u}| \leqslant c_1 \|\mathbf{u}\|_{H_2}; \quad \|\mathbf{u}\|_{W_4^1(S)} \leqslant c_2 \|\mathbf{u}\|_{H_2} \tag{2.14}$$

resulting from the second fundamental inequality for elliptic operators /13/ are used. The constants $c_{1,2}$ are independent of $\mathbf{u}$.

Lemmas 1 and 3 as a set denote that the operator B_Ω is a Fredholm operator /11/. It follows from Lemma 2 that small solutions $\mathbf{u} \in H_2$; $\partial\mathbf{u}/\partial\theta \neq 0$ in H for (2.1) and (2.6) should be sought only for values of the parameter γ close to γ_*. We shall seek small real solutions in the form

$$\mathbf{u} = \alpha\varphi_1 + \bar{\alpha}\varphi_2 + \mathbf{y}; \quad (\mathbf{y}, \Phi_i)_H = 0, \quad i = 1, 2 \tag{2.15}$$

and their corresponding angular frequencies in the form $\Omega = \Omega_0 + \mu$, where α and μ are unknown small parameters. It can be seen that $T_a\varphi_i \in N(B)$, $i = 1, 2$; $T_a\mathbf{y} \in H_2 \setminus N(B) = {}^\perp N(B)$ (${}^\perp N(B)$ is the orthogonal complement to $N(B)$ in H_2). Since the solution $\mathbf{u}$ is determined only to the accuracy of the transformation T_a, the parameter a of the family $\mathbf{u}_a = T_a\mathbf{u}$ can be fixed by the requirement $\alpha > 0$ and we can set in place of (2.15) (by analogy with /12/)

$$\mathbf{u} = \alpha\psi + \mathbf{y} \quad (\alpha > 0, \psi = \varphi_1 + \varphi_2) \tag{2.16}$$

Substituting (2.16) into (2.1) and taking account of (2.6), we obtain

$$B\mathbf{y} = -\mu\frac{\partial}{\partial\theta}(\alpha\psi + \mathbf{y}) - \sum_{s=1}^{\infty}\sum_{q=0}^{\infty}{}' \delta^q M_{sq}(\alpha\psi + \mathbf{y})^s \tag{2.17}$$

(the prime denotes that the term $M_{10}(\alpha\psi + \mathbf{y})$) is not present in the sum). Here P denotes the projector on $H \setminus N(B^*) = {}^\perp N(B^*)$

$$P\mathbf{u} = \mathbf{u} - (\mathbf{u}, \Phi_1)_H\varphi_1 - (\mathbf{u}, \Phi_2)_H\varphi_2 \tag{2.18}$$

Projecting (2.17) on to $N(B^*)$ and on to ${}^\perp N(B^*)$, we obtain

$$B\mathbf{y} = P\left\{-\mu\frac{\partial}{\partial\theta}(\alpha\psi + \mathbf{y}) - \sum_{s=1}^{\infty}\sum_{q=0}^{\infty}{}' \delta^q M_{sq}(\alpha\psi + \mathbf{y})^s\right\} \tag{2.19}$$

$$(I-P)\left\{-\mu\frac{\partial}{\partial\theta}(\alpha\psi+y)-\sum_{s=1}^{\infty}\sum_{q=0}^{\infty}{}'\delta^q M_{sq}(\alpha\psi+y)^s\right\}=0 \tag{2.20}$$

We shall consider B as an operator acting from $^{\perp}N(B)$ into $^{\perp}N(B^*)$. Then all the conditions of the theorem on implicit operators /11/ are satisfied for (2.19): for sufficiently small δ, α, μ, y, its right side is a continuous operator in $^{\perp}N(B^*)$ analytically dependent on δ, α, μ, y, while the operator B^{-1} exists and is bounded by virtue of Lemma 3. Consequently, in a sufficiently small neighbourhood of the point $y=0$, $\delta=\alpha=\mu=0$ a unique solution $y=y(\delta,\alpha,\mu)$, exists which can be found by the method of undetermined coefficients by substituting the following series into (2.19):

$$y=\sum_{s,q,m=0}^{\infty}y_{sqm}\delta^s\alpha^q\mu^m;\qquad y_{000}=0 \tag{2.21}$$

By such a method we obtain

$$y_{100}=y_{010}=y_{001}=y_{200}=y_{002}=y_{101}= \tag{2.22}$$
$$y_{011}=y_{300}=y_{003}=y_{201}=y_{102}=y_{012}=0$$

$$y_{020}=-B^{-1}PM_{20}\psi^2;\qquad y_{110}=-B^{-1}PM_{11}\psi$$
$$y_{030}=-B^{-1}P\left[M_{30}\psi^3+2M_{20}(\psi,y_{020})\right] \tag{2.23}$$
$$y_{210}=-B^{-1}P\left[M_{12}\psi+M_{11}y_{110}\right];\qquad y_{111}=-B^{-1}P\left[\partial y_{110}/\partial\theta\right]$$
$$y_{120}=-B^{-1}P\left[2M_{20}(\psi,y_{110})+M_{21}\psi^2+M_{11}y_{020}\right]$$

etc. Now substituting (2.21) into (2.20), we obtain a system of two bifurcation equations, each of which has the form

$$h(\delta,\alpha,\mu)\equiv\sum_{s,q,m=0}^{\infty}h_{sqm}\delta^s\alpha^q\mu^m=0;\qquad h_{000}=0 \tag{2.24}$$

Since real solutions of (2.1) are sought, the expression in the braces in (2.20) should be real and the bifurcation equations are equivalent. Discarding one and multiplying the other by -1, we obtain

$$\begin{aligned}-h(\delta,\alpha,\mu)\equiv{}&-ip\alpha\mu+(M_{11}\psi,\Phi_1)_H\,\delta\alpha+\\&(M_{12}\psi+M_{11}y_{110},\Phi_1)_H\,\delta^2\alpha+(M_{30}\psi^3+2M_{20}(\psi,y_{020}),\Phi_1)_H\,\alpha^3+\\&(\partial y_{110}/\partial\theta,\Phi_1)_H\,\delta\alpha\mu+\ldots=0\end{aligned} \tag{2.25}$$

where terms above the third power have been omitted.

Theorem 1. Let conditions 1 and 2 be satisfied, and let the following inequalities

$$\operatorname{Re}h_{110}\neq 0;\ \operatorname{Re}h_{030}\neq 0 \tag{2.26}$$

hold.

Then γ_* is a bifurcation point of two and only two solutions of the RSW type. These solutions exist for small $\delta>0$ $(\delta<0)$ if $\operatorname{Re}h_{110}\operatorname{Re}h_{030}<0$ $(\operatorname{Re}h_{110}\operatorname{Re}h_{030}>0)$, are specularly symmetric to each other, have p branches, and are analytic functions of the parameter $\sqrt{\delta}$ $(\sqrt{-\delta})$; the angular frequencies $\Omega_{1,2}=\pm(\Omega_0+\mu)$ correspond to them, where μ is an analytic function of μ. We hence have

$$\begin{aligned}\alpha&=(-\delta\operatorname{Re}h_{110}/\operatorname{Re}h_{030})^{1/2}+O(\delta)\\ \mu&=p^{-1}\delta\,\{\operatorname{Im}h_{110}-\operatorname{Re}h_{110}\operatorname{Im}h_{030}/\operatorname{Re}h_{030})+O(\delta^2)\end{aligned} \tag{2.27}$$

Proof. Note that h is an odd function of the variable α: the substitution $\theta\to\theta+p^{-1}\pi$, $\alpha\to-\alpha$ does not alter (2.19), while the left side of (2.20) transforms into its opposite number. Reducing (2.25) by $-\alpha$ and writing down the first-degree terms in δ, α^2, μ explicitly, we obtain

$$h_1(\delta,\alpha,\mu)\equiv ip\mu+h_{110}\delta+h_{030}\alpha^2+\ldots=0 \tag{2.28}$$

We set $\operatorname{Re}h_1=h_{1r}$, $\operatorname{Im}h_1=h_{1i}$, and we evaluate the Jacobian

$$\left.\frac{\partial(h_{1r},h_{1i})}{\partial(\alpha^2,\mu)}\right|_{\delta=\alpha=\mu=0}=-p\operatorname{Re}h_{030}\neq 0 \tag{2.29}$$

From (2.29) and the theorem on implicit functions it follows that for sufficiently small δ, Eq.(2.28) can be solved for α^2, μ: functions $\alpha^2=\xi(\delta)$ and $\mu=\eta(\delta)$ exist, analytic at the point $\delta=0$, transforming (2.28) into an identity, and uniquely defined by the requirement $\xi(0)=\eta(0)=0$. Furthermore, we obtain from (2.29)

$$\xi'(0)=\frac{\operatorname{Re}h_{110}}{\operatorname{Re}h_{030}};\qquad \eta'(0)=\frac{1}{p}\left(\operatorname{Im}h_{110}-\operatorname{Re}h_{110}\frac{\operatorname{Im}h_{030}}{\operatorname{Re}h_{030}}\right)$$

Hence, equalities (2.27) follow.

Small solutions of the RSW type are representable in the form of the series

$$\mathbf{u}_{1,2} = \alpha\boldsymbol{\psi}(r, \pm\theta) + \alpha^2\mathbf{y}_{020}(r, \pm\theta) + \delta\alpha\mathbf{y}_{110}(r, \pm\theta) + \ldots \tag{2.30}$$

all of whose non-zero terms, as can be seen by analyzing (2.22), (2.23), etc., are periodic in $\theta = \theta' - \Omega t$ with minimal common period $2\pi/p$. Therefore, they have p branches. The theorem is proved.

From condition 1 the inequalities (2.14), and the results of /14/ follow that small (in the norm of the space H_2) solutions of (2.1) are functions from $C^\infty(S)$.

3. Bifurcation of standing waves and radially symmetric time-periodic solutions. It is assumed below that $p \geqslant 0$. It will be proved that for $p > 0$ and when the conditions in Sect.1 are satisfied, a unique time-periodic solution, even in the angular coordinate - a standing wave /7, 8/, is bifurcated from the trivial solution of system (1.2) and (1.3). Because of the arbitrariness of the selection of the origin of the angular coordinates, this denotes generation of a system of one-parameter families of single-frequency time-periodic solutions (an invariant torus) in phase space. For $p = 0$ a unique time-periodic solution is bifurcated from the trivial solution of system (1.2), (1.3). This solution is radially symmetric, and possibly corresponds to the experimentally observed leading centre regime. Its asymptotic form, nature of bifurcation, and stability can be studied by methods developed in /12, 15/.

We reduce (1.2) to the form

$$\omega\, \partial\mathbf{u}/\partial\tau = D\Delta\mathbf{u} + F(\mathbf{u}, \gamma) \tag{3.1}$$

$$\partial\mathbf{u}(\tau, r, \theta')/\partial r\,|_{r=1} = 0 \tag{3.2}$$

$$\mathbf{u}(\tau, r, \theta' + 2\pi) = \mathbf{u}(\tau, r, \theta');\ \mathbf{u}(\tau + 2\pi, r, \theta') = \mathbf{u}(\tau, r, \theta')$$

by substituting $\omega t \to \tau$ (ω is the unknown cyclic frequency of the desired periodic solution).

Let Q_τ denote the set of complex-valued n-vector functions $\mathbf{u}$ defined in $[0, 2\pi] \times S$ and such that

1°) The function $\mathbf{u}$ is continuously differentiable with respect to τ and $\mathbf{u} \in Q$ for any fixed τ (see Sect.2);

2°) Conditions (3.2) are satisfied;

3°) If $p > 0$, the function $\mathbf{u}$ is even in θ'.

Let Q° is a subset of the set Q isolated by condition 3°; the Hilbert spaces H°, H_2° and H_τ°, $H_{2\tau}^\circ$ are closures of the sets Q° and Q_τ° in the metrics of (2.3) and

$$(\mathbf{u}, \mathbf{v})_{H_\tau^\circ} = \int_0^{2\pi} (\mathbf{u}, \mathbf{v})_{H^\circ}\, d\tau$$

$$(\mathbf{u}, \mathbf{v})_{H_{2\tau}^\circ} = \int_0^{2\pi} \left[\omega_0^2\left(\frac{d\mathbf{u}}{d\tau}, \frac{d\mathbf{v}}{d\tau}\right)_{H^\circ} + (\mathbf{u}, \mathbf{v})_{H_2^\circ}\right] d\tau$$

respectively. We introduce the operator $A: H_2^\circ \to H^\circ$ and its conjugate operator in H°, A^*: $H_2^\circ \to H^\circ$, by setting

$$A\mathbf{u} \equiv -D\Delta\mathbf{u} - M_{10}\mathbf{u};\quad A^*\mathbf{u} \equiv -D\Delta\mathbf{u} + M_{10}^T\mathbf{u}\ (\mathbf{u} \in H_2^\circ)$$

The assumptions of Sect.1 mean that the intersections of the spectra of the operators A and A^* with the imaginary axis consist of pairs of simple eigenvalues $\pm i\omega_0$. The eigenvector of the operator A corresponding to its eigenvalue $-i\omega_0$ is denoted by $\boldsymbol{\varphi}^\circ$, i.e.,

$$A\boldsymbol{\varphi}^\circ + i\omega_0\boldsymbol{\varphi}^\circ = 0;\quad \boldsymbol{\varphi}^\circ = \mathbf{g}_0 \cos p\theta' J_p(\varkappa_{pk} r)$$

while the eigenvector of the operator A^* corresponding to the eigenvalue $+i\omega_0$ is denoted by $\boldsymbol{\Phi}^\circ$, i.e.,

$$A^*\boldsymbol{\Phi}^\circ - i\omega_0\boldsymbol{\Phi}^\circ = 0;\ \boldsymbol{\Phi}^\circ = \mathbf{g}_0^T \cos p\theta' J_p(\varkappa_{pk} r)$$

Here

$$\mathbf{g}_0 = \begin{cases} \mathbf{g}, & p = 0 \\ \mathbf{g}\sqrt{2}, & p > 0 \end{cases};\quad \mathbf{g}_0^T = \begin{cases} \mathbf{g}^T, & p = 0 \\ \mathbf{g}^T\sqrt{2}, & p > 0 \end{cases}$$

The equation $(\boldsymbol{\varphi}^\circ, \boldsymbol{\Phi}^\circ)_{H^\circ} = 1$ holds.

We pose the problem of seeking small non-zero 2π-periodic solutions $\mathbf{u}$ of (3.1) in the Hilbert space $H_{2\tau}^\circ$. We call the number γ_* a cycle bifurcation point if one-parameter families ω_γ, $\mathbf{u}_\gamma : \mathbf{u}_\gamma \in H_{2\tau}^\circ$; $\mathbf{u}_\gamma \neq 0$ $(\gamma \neq \gamma_*)$, exist that are continuous in $\gamma \in \Gamma$ and satisfy (3.1), when γ runs through a certain interval, and have γ_* as their limit point, where $\omega_\gamma \to \omega_0$; $\mathbf{u}_\gamma \to 0$ in $H_{2\tau}^\circ$ for $\gamma \to \gamma_*$.

The solution of the above problem reduces to studying a bifurcation equation of the form

(2.24); the proofs needed for this, the auxiliary propositions, the reduction procedure, and the construction of the asymptotics of the small solutions are executed exactly as in /12/. We will limit oursevles solely to formulating the result.

We will use the following notation:

$$\mu = \omega - \omega_0;\ \psi^\circ = \varphi^\circ e^{i\tau} + \overline{\varphi^\circ} e^{-i\tau}$$
$$\mathbf{z}_0 = 2A^{-1}M_{20}(\varphi^\circ, \overline{\varphi^\circ})$$
$$\mathbf{z}_2 = (A + 2i\omega_0 I)^{-1} M_{20}(\varphi^\circ, \varphi^\circ)$$
$$f_{110} = (M_{11}\psi^\circ,\ \Phi^\circ e^{i\tau})_{H_\tau^\circ} = 2\pi (M_{11}\varphi^\circ,\ \Phi^\circ)_H$$
$$f_{030} = 2\pi (2M_{20}(\varphi^\circ, \mathbf{z}_0) + 2M_{20}(\overline{\varphi^\circ}, \mathbf{z}_2) + 3M_{30}(\varphi^\circ, \varphi^\circ, \overline{\varphi^\circ}), \Phi^\circ)_H$$

Theorem 2. Let conditions 1 and 2 be satisfied and the inequalities

$$\mathrm{Re}\, f_{110} \mathrm{Re}\, f_{030} < 0$$

hold.

Then γ_* is the bifurcation point of the single cycle $\mathbf{u} \in H_{2\tau}^\circ$ which exists for small $\delta > 0$ $(\delta < 0)$ if $\mathrm{Re}\, f_{110} \neq 0$; $\mathrm{Re}\, f_{030} \neq 0$ $(\mathrm{Re} f_{110} \mathrm{Re} f_{030} > 0)$, and is an analytic function of the parameter $\sqrt{\delta}$ $(\sqrt{-\delta})$; the quantitity μ is an analytic function of δ. Hence

$$\mathbf{u} = (-\delta\, \mathrm{Re}\, f_{110}/\mathrm{Re}\, f_{030})^{1/2}\psi^\circ + O(\delta)$$
$$\mu = (2\pi)^{-1}\delta\, (\mathrm{Im}\, f_{110} - \mathrm{Re}\, f_{110} \mathrm{Im}\, f_{030}/\mathrm{Re}\, f_{030}) + O(\delta^2)$$

It can be seen by direct calculations that when $p > 0$ the cycle being generated at any instant is a periodic function in θ' with period $2\pi/p$, and when $p = 0$ is a radially symmetric function (independent of θ').

Example. Consider the system of equations

$$\begin{aligned} \partial u_1/\partial t &= R^2(-u_1 + 3\sqrt{\varepsilon} u_3) + \Delta u_1 \\ \partial u_2/\partial t &= R^2(-\varepsilon u_2 - \beta u_3 + c u_2^3) + \varepsilon^2 \Delta u_2 \\ \partial u_3/\partial t &= R^2(3\sqrt{\varepsilon} u_1 + \beta u_2 - \varepsilon u_3) + \varepsilon^2 \Delta u_3 \\ &\varepsilon, \beta \geqslant 0; \quad \varepsilon \ll 1; \quad c \neq 0 \end{aligned} \tag{3.3}$$

in a circle S with impermeability conditions on the boundary. Here

$$\gamma = R^2; \quad M_1(\gamma) = \begin{Vmatrix} -1 & 0 & 3\sqrt{\varepsilon} \\ 0 & -\varepsilon & -\beta \\ 3\sqrt{\varepsilon} & \beta & -\varepsilon \end{Vmatrix} R^2; \quad D = \mathrm{diag}(1, \varepsilon^2, \varepsilon^2)$$

The concentrated system corresponding to (3.3) describes a non-linear oscillator with friction and a regulator in the positive feedback loop. In a distributed system the "substance"-regulator diffusion coefficient greatly exceeds the diffusion coefficients of the oscillator components.

Let $\chi > 0$ be a real parameter. Consider the matrix $L(\chi) = -\chi D + M_1(\gamma)/R^2$. We call the minimal closed interval $[\chi_1, \chi_2]$ of values χ $(0 < \chi_1 < \chi_2)$ outside of which the matrix $L(\chi)$ is stable (i.e., its spectrum lies in theleft half-plane) the instability interval. We set $\beta = \sqrt{5}$. We have for the Hurwitz determinants of the characteristic equation of the matrix $L(\chi)$

$$G_P > 0,\ G_Q > 0,\ G_R > 0,\ G_P G_Q - G_R = \varepsilon f^\circ(\varepsilon, \chi)$$

Here $f^\circ(\varepsilon, \chi)$ is a third power polynomial in χ. The roots $\chi_1 = 1, \chi_2 = 3/2$ of the polynomial $f^\circ(0, \chi)$ vary slightly for $\varepsilon > 0$ according to the theorem on implicit functions: $\chi_1 \approx 1 + 17\varepsilon + O(\varepsilon^2)$, $\chi_2 \approx 1.5 - 34.75\varepsilon + O(\varepsilon^2)$. Moreover, for $\varepsilon > 0$ there is a negative root $\chi_0 < 0$.

For sufficiently small R the trivial solution of system (3.3) is asympotically stable. As R grows it becomes unstable four times: for

$$R^2 = R_0^2 = \varkappa_{11}^2/\chi_2,\ R^2 = R_2^2 = \varkappa_{31}^2/\chi_2,\ R^2 = R_4^2 = \varkappa_{02}^2/\chi_2,\ R^2 = R_6^2 = \varkappa_{41}^2/\chi_2$$

and again becomes stable three times: for

$$R^2 = R_1^2 = \varkappa_{11}^2/\chi_1,\ R^2 = R_3^2 = \varkappa_{21}^2/\chi_1,\ R^2 = R_5^2 = \varkappa_{31}^2/\chi_1;\ 0 < R_0 < R_1 < R_2 < R_3 < R_4 < R_5 < R_6.$$

For $R > R_6$ stability is no longer acquired. For losses of stability at the critical times, the eigenvalues $\pm i\omega_0$ of problem (1.6) are generated, respectively, by the eigenvalues $\varkappa_{11}^2, \varkappa_{21}^2, \varkappa_{02}^2$ and $\varkappa_{41}^2$ of problem (1.7). An increase in the number of losses of stability to any previously assigned number can be attained be selecting $\beta \neq \sqrt{5}$.

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ON SAINT-VENANT TYPE CONDITIONS IN THE THEORY OF PIEZOELASTIC SHELLS*

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Saint-Venant type conditions extended to piezoelasticity are formulated. It is shown that one electrical condition is added to the well-known Saint-Venant mechanical conditions for piezoceramic shells with non-electrodized face surfaces. The Saint-Venant conditions accepted in elasticity theory remain true for shells with electrodized face surfaces.

The complete state of stress and strain of a non-electric elastic shell is comprised of a deeply penetrating internal state of stress and strain described by the equations of shell theory, and of boundary layers localized near the edges. In formulating the boundary conditions for the internal state of stress and strain and the boundary layers, an important part is played by the Saint-Venant principle /1/, which is as follows as applied to elastic shells: if stresses are given arbitrarily on the edge of a shell, then non-selfequilibrated edge effects will generate a deeply penetrating solution and should be taken into account when analysing the state of stress and strain, while the part of the edge load not selfequilibrated over the thickness will cause a stress and strain state that will damp rapidly at the edge and is taken into account in analysing the boundary layer.

In the case of piezoelastic shells, both electrical and mechanical quantities occur in the complete system of equations. Consequently, the question arises of what conditions of Saint-Venant type should the mechanical and electrical edge load satisfy. To answer this question, following /2/, we find a solution of the boundary layer problems and we clarify, in passing, what requirements the edge load should be subjected to in order for the boundary layer solution to have the necessary damping. That part of the load which does not satisfy these conditions should be taken into account in analysing the internal electroelastic state of the shell.

1. We select a system of tri-orthogonal coordinates as follows: curvilinear coordinates α_1 and α_2-lines of curvature of the middle surface, and γ-lines orthogonal to them.

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